A meta-sequence is defined to be a sequence consisting of $k$ sub-sequences, with each sub-sequence containing $k$ equally spaced elements. Two methods of generating prime magic squares, the Agrippe method and the knight’s move method, are detailed and contrasted. The knight’s move method is shown to be superior for certain cases, and is shown to be useful for generating magic squares from meta-sequences of numbers. It is conjectured that arbitrarily long meta-sequences of prime numbers exist, and may be used to generate prime magic squares. A $7 \times 7$ prime magic square is presented.
**DEFINITION:** Arithmetic Sequence: A sequence of natural numbers of the form \( a + nd \), where \( a \) and \( d \) are fixed, and \( n \) is the cardinal position of the element in the sequence. For example, for the arithmetic sequence \( S_i = \{s_0, s_1, s_2, s_3\} = \{3, 5, 7, 9\} \), we have \( a = 3 \), and \( d = 2 \). \( a \) is the base of the sequence, and \( d \) is the distance between elements of the sequence. The cardinality of a sequence may also be referred to as its length.

**DEFINITION:** Meta-sequence: A sequence composed of arithmetic sequences (called sub-sequences) of equal length and distance. The number of sub-sequences in a meta-sequence is equal to the number of elements in each sub-sequence. \( M = \{S_0, S_1, \ldots, S_k\} \), where \( S_i = \{s_i + 0d, s_i + 1d, \ldots, s_i + (k - 1)d\} \). Notice that \( S_i \) is a sub-sequence, and \( s_i \) is an element of that sub-sequence. \( s_i \) is different for each element of the meta-sequence, but \( d \) is constant for all of them.

**NOTE:** A sequence with cardinality \( n^2 \) may be considered a meta-sequence with \( n \) elements.

**DEFINITION:** Magic Square: An \( n \times n \) matrix which has the same sum (called the magic sum) for every row, column, and full diagonal.

**DEFINITION:** Prime Magic Square: A magic square comprised entirely of (unique) prime numbers.

Magic squares have been present in mathematics for thousands of years, and a treatment of their variations can be found in many recreational mathematics texts. This paper addresses the topic of \( 2n + 1 \) by \( 2n + 1 \) magic squares in general, and that of prime magic squares in particular.

As described in the definition, a magic square is comprised of a matrix which has a sequence of numbers, say from 1 to 9, placed within smaller square which subdivide it. Below is a typical magic square for the numbers 1 through 9.

\[
\begin{array}{ccc}
4 & 9 & 2 \\
3 & 5 & 7 \\
8 & 1 & 6
\end{array}
\]

The magic sum for this magic square is 15.

There are a plethora of different algorithms for generating magic squares. Two of the most common are the diagonal method described by Agrippe, and the knight’s move method. The Agrippe diagonal method is used for making odd squares (squares with an odd side length) of any size from a single sequence. A basic outline of the Agrippe diagonal algorithm is as follows:
The Agrippe Diagonal Method

1) Place the least element of the sequence in the square directly beneath the center sub-square.

2) Move diagonally downward to the right, (wrapping around where necessary), filling in the next box, if empty, with the next sequence element.

3) When a full box is encountered, move diagonally one box down and to the left from the full box, and proceed normally.

In this manner the entire square will be filled in, and the result will be a magic square.

Agrippe’s diagonal method creates magic squares with several predictable properties. Obviously, the smallest element in the sequence will always be directly beneath the center of the square. Less obviously, the center square will contain the median of the sequence, and the square directly above the center will contain the greatest element of the sequence.

Agrippe’s diagonal method suffices for creation of magic squares from a single sequence, and from the special case of a degenerate meta-sequence, that is, a meta-sequence which is also a sequence. However, in the case that a meta-sequence is not also a sequence, the algorithm described previously will fail. Thus, if we desire to create a magic square of prime numbers, we must find a sequence which is \(3 \times 3 = 9\) long, or \(5 \times 5 = 25\) long, and so forth. Since the longest arithmetic sequence of primes yet discovered is only \(19\) elements long, and begins in the billions, and since Agrippe’s algorithm yields only a single magic square for a given sequence, the possibilities using known sequences are quite limited.

The knight’s move algorithm, however, will generate a magic square from a meta-sequence. Thus, if we can find a meta-sequence with seven elements (see example in Appendix A), we have more options available to us. Here is the knight’s move algorithm.
Knight’s Move Algorithm

Given: for \( n \geq 2, z = 2 \cdot n + 1, 3 \} \}, A meta-sequence \( M = \{S_0, S_1, ..., S_z\} \).

1) Begin at any square, with \( i = j = 0 \).

2) Place \((S_i)_j\) in the current square.

3) Move two squares to the right, one square upwards

4) Increment \( j \). If \( j < z \), goto 2).

5) Move two squares downward, one square to the left.

6) Increment \( i \). set \( j = 0 \). If \( i < z \), goto 2).

One especially nice aspect of the knight’s move algorithm is the ability to “begin” at any square of the matrix, and to use the sub-sequences of the meta-sequence in ANY order, so long as the order of the elements in the sub-sequence is preserved! Thus, for a meta-sequence of cardinality 5, since there are \( 5^2 \) possible starting positions, and \( 4 \cdot 3 \cdot 2 \cdot 1 \) possible orderings of the 5 sub-sequences (don’t forget the first is considered anchored since we are counting it as being able to start anywhere in the magic square), there are \( 5 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 5! = 600 \) different magic squares! For a meta-sequence of cardinality 11, there would be \( 11 \cdot 11! \) different magic squares, or 439,084,800 combinations. That’s right, you can throw away the crossword puzzle book, there’s a new calling in life!

A few practice runs with arbitrarily selected meta-sequences should be enough to convince you of the validity of the knight’s move method. However, the reason it works is worth examining.

As mentioned earlier, a meta-sequence is comprised of \( z \) sub-sequences, with each sub-sequence containing \( z \) elements. When the knight’s move algorithm is applied, this will generate a \( z \times z \) magic square. Our magic sum – the sum of any row, column, or diagonal – is the mean of the entire meta-sequence.

Let us take the case of a meta-sequence of cardinality 5.

\[
M = \{ S_0 = \{a_0, a_1, a_2, a_3, a_4\}, \\
S_1 = \{b_0, b_1, b_2, b_3, b_4\}, \\
S_2 = \{c_0, c_1, c_2, c_3, c_4\}, \\
S_3 = \{e_0, e_1, e_2, e_3, e_4\}, \\
S_4 = \{f_0, f_1, f_2, f_3, f_4\} \}
\]
A magic square generated with the knight’s move algorithm, starting in the upper left hand corner, would appear as pictured below. Note that this starting position, because of the nature of the meta-sequence, does not cause loss of generality with respect to starting position and sub-sequence order.

<table>
<thead>
<tr>
<th>a₀</th>
<th>c₄</th>
<th>f₃</th>
<th>b₂</th>
<th>e₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>f₂</td>
<td>b₁</td>
<td>e₀</td>
<td>a₄</td>
<td>c₃</td>
</tr>
<tr>
<td>e₄</td>
<td>a₃</td>
<td>c₂</td>
<td>f₁</td>
<td>b₀</td>
</tr>
<tr>
<td>e₁</td>
<td>f₀</td>
<td>b₄</td>
<td>e₃</td>
<td>a₂</td>
</tr>
<tr>
<td>b₃</td>
<td>e₂</td>
<td>a₁</td>
<td>e₀</td>
<td>f₄</td>
</tr>
</tbody>
</table>

Examination of this square reveals that there is exactly one element from each sub-sequence in each row and column, and on each full diagonal. Furthermore, no two elements in a given row, column, or diagonal have the same position in their respective sub-sequences, as denoted by the subscript. Recall that since $M$ is a meta-sequence, $S₀ = \{a + 0d, a + 1d, a + 2d, \ldots\}$ where $d$ is a constant, the distance for ALL the sub-sequences. Since for $\{a₀ = a + nd, 0 \leq a < \infty\}$, $a$ is present in all sums, we may remove $a$ from the square without changing the difference between sums. This principal may be extended to all the elements in the magic square, leaving us with

<table>
<thead>
<tr>
<th>0d</th>
<th>4d</th>
<th>3d</th>
<th>2d</th>
<th>1d</th>
</tr>
</thead>
<tbody>
<tr>
<td>2d</td>
<td>1d</td>
<td>0d</td>
<td>4d</td>
<td>3d</td>
</tr>
<tr>
<td>4d</td>
<td>3d</td>
<td>2d</td>
<td>1d</td>
<td>0d</td>
</tr>
<tr>
<td>1d</td>
<td>0d</td>
<td>4d</td>
<td>3d</td>
<td>2d</td>
</tr>
<tr>
<td>3d</td>
<td>2d</td>
<td>1d</td>
<td>0d</td>
<td>4d</td>
</tr>
</tbody>
</table>

Again, since $d$ is present in each box of the square, we may remove it to the outside of the magic square as a multiplier.

<table>
<thead>
<tr>
<th>0</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

$d \times$

It is now obvious that the sum of each row, column, and main diagonal is identical, and so we have a genuine magic square! This principal applies to ALL odd magic squares generated by the knight’s move algorithm, subject to the algorithm’s constraints.

Why doesn’t the knight’s move algorithm work for side lengths divisible by three? An excellent question! In those cases, the row and column sums are all equal to the magic sum. However, when it comes time to add up the diagonals, the diagonal totals are different from the magic sum.
Let us take a closer look at how a sub-sequence gets distributed in the magic square by examining $7 \times 7$ and $9 \times 9$ examples with the first subsequence entered, starting from the bottom left corner.

\[
\begin{array}{cccc}
   & a_3 & & \\
 a_4 & & a_5 & \\
 & a_2 & & a_6 \\
 a_1 & & & \\
\end{array}
\]

Notice that in the $7 \times 7$ square only one element is on the diagonal, while in the $9 \times 9$ square, three elements are on the diagonal. The reason for this is difficult to find, but easy to see once it’s revealed.

**THEOREM:** A magic square cannot be made using the knight’s move algorithm for a meta-sequence if three divides the cardinality of the meta-sequence.

**Proof:** Think of a would-be magic square as a cartesian grid with the origin $(0,0)$ contained in the lower left-hand box, and the square diagonally opposite it being $(z - 1, z - 1)$. One of the diagonals extends from $(0, z - 1)$ to $(z - 1, 0)$, and is composed of every point $(s,t)$ where $s + t = z - 1$. Suppose we’re dealing with a grid where $z = 3n$, and a magic square exists for this grid. We know a single move will change our position $(s, t)$ by $(+2, +1)$. We also know that we have a total of $(z - 1)$ moves, not including returning to our starting position of $(z - 1, 0)$. We are looking for all $i, 0 \leq i \leq (z - 1)$, such that $((z - 1) + 2i) + (0 + i) = z - 1$. This simplifies as follows (remember, we’re working in MOD $z$):

\[
(z - 1) + 2i + 0 + i = z - 1 \\
3i = 0
\]

Obviously, $0$ is a solution. As well, if $3$ divides $z$, there will be two other solutions, $n$ and $2n$. Since we can have at most one element from a subsequence on a diagonal, we have a contradiction, $QED$. 

Now that the basics of constructing magic squares from meta-sequences have been covered, we can move on to the method of generating meta-sequences of prime numbers.

Part 2: Arithmetic Sequences of Prime Numbers

Prime magic squares have been constructed, but the examples published before this paper have been quite small, $3 \times 3$ being the usual limit. Since the longest arithmetic sequence of primes yet discovered contains 19 elements, this would limit the size of the magic squares we could compose with the two methods discussed in this paper to $3 \times 3$. However, if we could find meta-sequences of primes, we could construct a larger square. This is not as impossible as it sounds: we have found meta-sequences of cardinality 7. Unfortunately, the time required to find a meta-sequence increases exponentially with respect to the cardinality of the meta-sequence. With a modicum of computer time (several thousand CPU hours on a relatively fast mini-computer), it should be possible to find prime meta-sequences of cardinality 11.

How do we know such meta-sequences exist?

**CONJECTURE:** There exist meta-sequences of cardinality $k$ for finite $k$.

This conjecture is based upon the following theorem, and examples up to $k = 9$. Dirichlet [1837] *If $d \geq 2$ and $a \neq 0$ are integers that are relatively prime, then the arithmetic progression $a, a + d, a + 2d, ...$ contains infinitely many primes.*

Clearly, if $a$ is a prime number, we are guaranteed of generating a sequence containing an infinite number of primes for all values of $d$. We conjecture that it is not unreasonable to expect short sub-sequences within this infinite sequence to be composed of primes.

One difficulty with this conjecture is the size of the numbers involved in the meta-sequences. It was shown by Cantor that for an arithmetic sequence of $n$ primes, the distance $d$ must be divisible by the product of all primes less than or equal to $d$ (unless the sequence begins with $n$ in the event $n$ is a prime number). In other words, a sequence of seven primes must have a distance $d = (2 \cdot 3 \cdot 5 \cdot 7) \cdot k$, except for the special case already noted. While this makes the search for large sequences easier in terms of primes which need to be checked, the direct implication is that the distance for larger sequences is ENORMOUS: for a sequence of 23 primes, excepting the case of a sequence beginning with 23, if it exists, you would have a minimum distance of 223,092,870 between elements, for a minimum sequence span of just under five billion! Even on a supercomputer, the calculations required to find these sequences would be prohibitive.

Assuming this conjecture is correct, it is possible to make arbitrarily large prime magic squares, with the knight’s move method enabling us to generate significantly larger squares per unit of computer time than Agrippé’s method. An example of a $7 \times 7$ sequence prime magic square is given below. If you are interested in the source code used to verify the magic squares and to find the meta-sequences, do not hesitate to contact either of the authors.

For more information about all aspects of prime numbers *The Book of Prime Number*
Records by Paulo Ribenboim is an excellent source.

Keywords: Magic Squares; Numbers, prime; mathematics, recreational.